

Quantum mechanics from invariance laws

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Quantum mechanics is an extremely successful theory of nature and yet it has resisted all attempts to date to have an intuitive axiomatization. In contrast, special theory of relativity is well understood and is rooted into natural or experimentally justified postulates. Here we show an axiomatization approach to quantum mechanics which is very similar with how special theory of relativity can be derived. The core idea is that of composing two systems and the fact that the composed system should have an invariant description in terms of dynamics. This leads to a Lie-Jordan algebraic formulation of quantum mechanics which can be converted into the usual Hilbert space formalism by the standard GNS construction. The starting assumptions are minimal: the existence of time and that of a configuration space which supports a tensor product as a way to compose two physical systems into a larger one.

Quantum mechanics is a very unintuitive theory: it predicts only probabilistic outcomes but it supposes to be the “whole story” [1], it exhibits correlations between separable systems which cannot be explained by “local realistic” means [2], it is based on an abstract formalism involving hermitean operators and complex vector spaces far removed from the immediate reality we experience every day. Also it is not even clear what do we mean by “quantum” [3].

In contrast, special theory of relativity is well understood and is rooted in two simple postulates: the laws of nature are invariant under changes in inertial frames of reference and the principle of invariant light speed. The first postulate of special theory of relativity tells us that there is no absolute reference frame and there are no distinguished speeds (except for the speed of light) and this is easy to understand. The second postulate (and its consequences) is harder to understand and is ultimately justified by experimental evidence (starting with the Michelson - Morley experiment [4]).

It is the aim of this paper to derive quantum mechanics in a very similar way with how special theory of relativity is obtained.

Special theory of relativity has a kinematical foundation, but emphasizing this fact obscures a larger point that it is based on a specific invariance of the laws of nature. In particular, special theory of relativity uses only one kind of invariance, related of inertial reference frames. Figure 1 presents one possible line of argument for deriving special theory of relativity.

In contrast, quantum mechanics can be derived from several other invariances and from natural or experimentally justified postulates (see Figure 2).

When talking about dynamics, we need to allow interaction between any two physical systems. The critical element is the invariance of dynamics under tensor composition of two subsystems. Composability was originally discovered in the 1970s by Emile Grignani and Aage Petersen [5] as the necessary ingredient of understanding classical

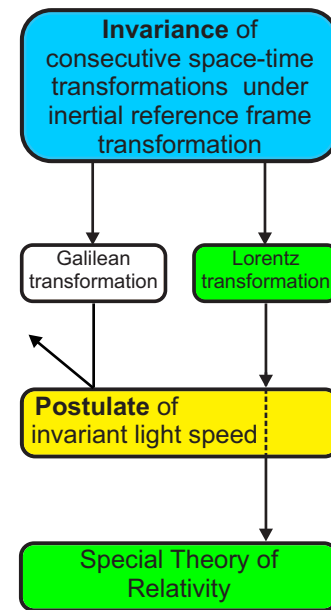


FIG. 1. Deriving special theory of relativity line of argument.

and quantum mechanics in a single mathematical formalism. The original motivation was a belief by Bohr (as reported by his personal assistant Aage Petersen) that the correspondence principle has more to reveal. This groundbreaking work resulted in what is now called a Lie-Jordan algebra [17] which is the foundation of the C^* algebraic formulation of quantum mechanics.

The Lie-Jordan algebra requires a norm and an important observation is that this norm is unique [6] given the spectral distance. Therefore to derive quantum mechanics we have to derive the necessity of the Jordan algebra of observables and the Lie algebra of generators together with their compatibility condition. From the Lie-Jordan structure, the usual Hilbert space formulation is obtained by the standard GNS construction [7].

In a Lie-Jordan approach to quantum mechanics, the starting point is the existence of the two products, one de-

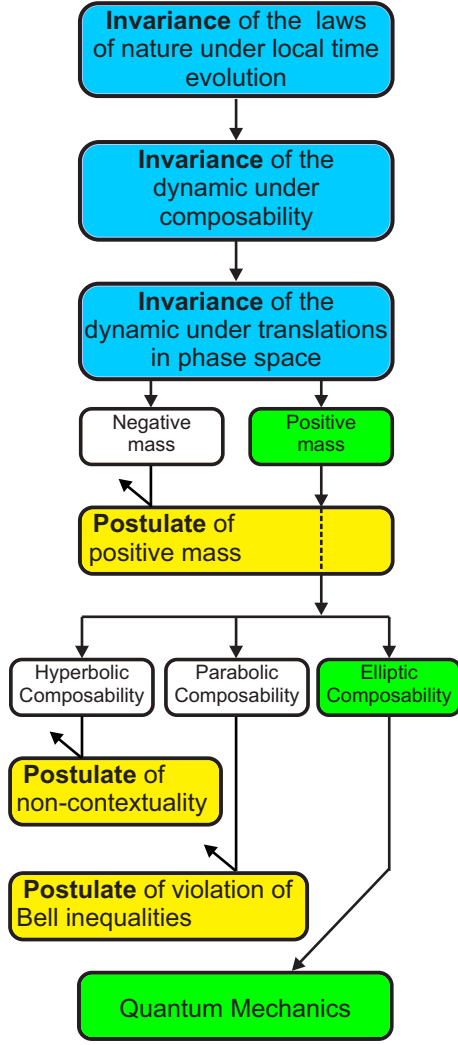


FIG. 2. Deriving quantum mechanics line of argument.

describing the generators and the other one the observables a duality known as “the equivalence of observables and generators” [8] or “dynamic correspondence” [9]. However, demanding the two products is too strong a requirement and we can start from much milder assumptions arriving at them.

The starting point is requiring the existence of time and a configuration space understood as a manifold endowed with a product with unit called a tensor product which captures the core idea of allowing interaction between two physical systems. It is not necessary to consider only manifolds, as long as one can define a cotangent space which is the natural place of Hamiltonian formalism.

For the sake of simplicity we will discuss only the case of configuration space in the form of a manifold M and $C^\infty(M)$ functions over this manifold. At a point $p \in M$ one can define a tangent space $T_p M$ and a cotangent space $T_p^* M$. We will also assume that there are some C^∞

functions for which there is a way to generate a vector field out of them, and from now on we will restrict the domain of discussion only for those functions.

Let the time evolution be represented by a one parameter group of transformations ϕ defined as follows:

$$\phi : M \times \mathbb{R} \rightarrow M \quad (1)$$

with $\phi(x, 0) = x$ and $\phi(\phi(x, t), s) = \phi(x, t + s)$.

Suppose that there is an unspecified family of local operations $\{\circ\}$ which describe the laws of nature. Introducing the notation: $\phi_t(x) \equiv \phi(x, t)$, the invariance of the laws of nature under time evolution reads:

$$(g \circ h)(\phi_{\Delta t}(p)) = g(\phi_{\Delta t}(p)) \circ h(\phi_{\Delta t}(p)) \quad (2)$$

with $p \in M$ a point in the manifold M and g, h , functions defined in the neighborhood of p . In other words, we demand the existence of a universal local morphism.

If X_f^i is a vector field arising out of a function f , define an operator \mathcal{T}_{f_ϵ} :

$$\mathcal{T}_{f_\epsilon} = I + \epsilon X_f^i \frac{\partial}{\partial u^i} \quad (3)$$

with I the identity operator and u^i the coordinate set in a local \mathbb{R}^n chart covering the point p .

Because f is in one to one correspondence with X_f^i there exists an operation $\rho \in \{\circ\}$ such that we can introduce $\mathcal{T}_{f_\epsilon} = (I + \epsilon f \rho)$. Then we can derive to first order in ϵ a left and a right Leibniz identity:

$$f \rho(g \circ h) = (f \rho g) \circ h + g \circ (f \rho h) \quad (4)$$

$$(g \circ h) \rho f = (g \rho f) \circ h + g \circ (h \rho f) \quad (5)$$

One important property to notice is ρ has no identity element such that $1 \rho f = f \rho 1 = f$. This is a simple consequence of the Leibniz identity when applied to ρ itself. Also for any f , $f \rho 1 = 1 \rho f = 0$. To put it in context, ρ will be later on shown to be the usual commutator in quantum mechanics (or the Poisson bracket in classical mechanics).

When talking about dynamics, it is usual to consider the interaction of two systems. The key property of a configuration space of many systems is the ability to concatenate the sub-systems using a tensor product. In mathematical language, a configuration space endowed with a tensor product forms a category.

Inspired by Grgin groundbreaking work [5] we can introduce a composability category $\mathcal{U} = \mathcal{U}(\otimes, \mathbb{R}, \rho, \dots)$ which respects the existence of a unit element, the real numbers field \mathbb{R} (understood as the constant functions set), meaning $\mathcal{U} \otimes \mathbb{R} = \mathcal{U} = \mathbb{R} \otimes \mathcal{U}$.

First it can be shown that the product ρ is not enough and this demands the existence of a secondary product $\theta \in \{\circ\}$. The invariance of the laws of nature under composability demands that a bipartite product ρ_{12} should be built out of the products listed in the composability category. If only a product ρ exists in the composability category, it must be trivial due to Leibnitz identity. Only by adding another product θ we can construct non-trivial mathematical structures.

The most general way to construct the products ρ and θ in a bipartite system is as follows:

$$(f_1 \otimes f_2)\rho_{12}(g_1 \otimes g_2) = a(f_1\rho g_1) \otimes (f_2\rho g_2) + b(f_1\rho g_1) \otimes (f_2\theta g_2) + c(f_1\theta g_1) \otimes (f_2\rho g_2) + d(f_1\theta g_1) \otimes (f_2\theta g_2) \quad (6)$$

$$(f_1 \otimes f_2)\theta_{12}(g_1 \otimes g_2) = x(f_1\rho g_1) \otimes (f_2\rho g_2) + y(f_1\rho g_1) \otimes (f_2\theta g_2) + z(f_1\theta g_1) \otimes (f_2\rho g_2) + w(f_1\theta g_1) \otimes (f_2\theta g_2) \quad (7)$$

with f_1, f_2, g_1, g_2 arbitrary functions over the manifold M at a point p .

Then we can use the existence of the unit of the composability category to determine the coefficients b, c, d, y, z, w . If we normalize the definition of product θ such that $1\theta 1 = 1$, we can show that $d = y = z = 0$ and $b = c = w = 1$. Applying the Leibnitz identity on the bipartite products demands $a = 0$. Hence the bipartite products in the shorthand notation are:

$$\begin{aligned} \rho_{12} &= \rho_1\theta_2 + \theta_1\rho_2 \\ \theta_{12} &= \theta_1\theta_2 + x\rho_1\rho_2 \end{aligned} \quad (8)$$

We can now observe that if ρ is a skew-symmetric product and θ is a symmetric product the symmetry and skew-symmetry is maintained under composition. This suggests decomposing ρ and θ as follows:

$$\begin{aligned} \rho &= \alpha + \tau \\ \theta &= \sigma + \pi \end{aligned} \quad (9)$$

with:

$$\begin{aligned} f\alpha g &= (f\rho g - g\rho f)/2 \\ f\tau g &= (f\rho g + g\rho f)/2 \\ f\sigma g &= (f\theta g + g\theta f)/2 \\ f\pi g &= (f\theta g - g\theta f)/2 \end{aligned} \quad (10)$$

We can form the most general odd and even bipartite products and using the identity element of the composability category, we can show that the following general relationships hold:

$$\sigma_{12} = \sigma_1\sigma_2 + x\tau_1\tau_2 + \pi_1\pi_2 + x\alpha_1\alpha_2 \quad (11)$$

$$\alpha_{12} = \pi_1\tau_2 + \tau_1\pi_2 + \alpha_1\sigma_2 + \sigma_1\alpha_2$$

$$\tau_{12} = \sigma_1\tau_2 + \tau_1\sigma_2 + \pi_1\alpha_2 + \alpha_1\pi_2$$

$$\pi_{12} = \pi_1\sigma_2 + \sigma_1\pi_2 + x\alpha_1\tau_2 + x\tau_1\alpha_2$$

We should also investigate what happens when one of the products in the above equations is identically zero. There are only two cases allowed, $\tau \equiv 0$ or $\alpha \equiv 0$ and the equations above reduces to either:

$$\begin{aligned} \sigma_{12} &= \sigma_1\sigma_2 + x\alpha_1\alpha_2 \\ \alpha_{12} &= \alpha_1\sigma_2 + \sigma_1\alpha_2 \end{aligned} \quad (12)$$

or

$$\begin{aligned} \sigma_{12} &= \sigma_1\sigma_2 + x\tau_1\tau_2 \\ \tau_{12} &= \tau_1\sigma_2 + \sigma_1\tau_2 \end{aligned} \quad (13)$$

with $x = -1, 0, 1$.

So far the product ρ is undefined. We know it is a derivation because it obeys the Leibniz identity, meaning it forms a Loday algebra [10], but it can be further constrained when considering a principle similar with the principle of relativity. Special relativity shows there are no absolute reference frames. We can generalize this relativity principle and demand no distinguished points on the configuration space manifold M meaning the laws of nature are invariant under configuration space translations. Suppose also that in the tangent bundle the evolution equations is described in general only by a single first order differential equation. The dynamics in this case is trivial and we eliminate this case from further consideration. Alternatively, this case can be eliminated by demanding the freedom to choose initial conditions.

In general there is no natural way to identify vectors with co-vectors, but given a function f , one can naturally construct df and also we had previously assumed the existence of a map to obtain a vector field X_f out of any function f . Therefore given this assumption, there must exists a vectors to co-vectors map realized by what can later be shown to be the Lagrangean L : $\dot{x}\partial/\partial x \rightarrow (\partial L/\partial \dot{x})dx$ which allows the usual definition of p : $p(x, \dot{x}) = \partial L/\partial \dot{x}$.

To obtain the most general bracket, it is helpful switch the discussion from manifolds to affine varieties and consider the space \mathbb{F} of polynomial functions vanishing on T^*M . Assuming that the dimension of M is d , we can define a bracket $\{\cdot, \cdot\}$ on $\mathbb{F}[x_1, \dots, x_{2d}]$ in the following way:

$$\{F, G\} = \sum_{i,j=1}^{2d} \{x_i, x_j\} \frac{\partial F}{\partial x_i} \frac{\partial G}{\partial x_j} \quad (14)$$

with $F, G \in \mathbb{F}[x_1, \dots, x_{2d}]$.

This bracket is the most general way to construct a product ρ : $F\rho G = \{F, G\}$ and the proof is by induction using the standard argument that two biderivations of some commutative associative algebra $\mathcal{A} = \mathbb{F}$ are equal as soon as they agree on a system of generators for \mathcal{A} [11].

In the one dimensional cotangent space case, the products we need to consider are: $x\rho x$, $x\rho p$, $p\rho x$, and $p\rho p$ with x the coordinate at point P , and p the canonical momentum coordinate.

Because we want the dynamic to be independent of any particular (distinguished) point P , $x\rho x = p\rho p = 0$. Also for the remaining two products, they both must be nontrivial if one wants to avoid having a single first order linear equations. Therefore (after a normalization of the coordinates), the only two cases are:

$$\begin{aligned} \{x_i, p_i\} &= x_i\rho p_i = 1 \\ \{p_i, x_i\} &= p_i\rho x_i = -1 \end{aligned} \quad (15)$$

which is the usual Poisson bracket or

$$\begin{aligned} \{x_i, p_i\} &= x_i\rho p_i = 1 \\ \{p_i, x_i\} &= p_i\rho x_i = 1 \end{aligned} \quad (16)$$

It is clear that to characterize the product ρ , its symmetry property is enough and the general composability case of Eq. (11) reduces to either Eq. (12) or Eq. (13).

The symmetric composability case $\rho = \tau$ gives rise in the classical case ($x = 0$) to a dynamic where the energy is bounded from above (the particles tend to move uphill, or equivalently the mass is negative).

If we postulate the necessity of positive mass we are compelled to eliminate this kind of composability from consideration and conclude that the only acceptable case is described by Eq. (12).

For the composability of Eq. (12), Jacobi identity $f\alpha(gah) + h\alpha(fag) + g\alpha(haf) = 0$ is trivially satisfied as a consequence of the Leibniz identity and the skew-symmetry of product α .

Let us summarize what we have derived up to this point. First, there is a skew-symmetric bilinear product α which obeys both a Leibnitz identity and a Jacobi identity. As such it forms a Lie algebra. Then there is a symmetric bilinear product σ and a universal parameter x which is a constant of nature. The normalized parameter x can be -1 , 0 , or $+1$ corresponding to elliptic, parabolic, or hyperbolic composability classes. In quantum mechanics case $x = -\hbar^2/4$ and the fact that the Planck constant is invariant is a non-trivial fact due to composability [12]. A side-effect of the invariance of the Planck constant as a consequence of composability is the impossibility to have a consistent theory of mixed classical-quantum world because classical and quantum mechanics belong in disjoint composability classes [12].

From $\sigma_{12} = \sigma_1\sigma_2 + x\alpha_1\alpha_2$ it is easy to see that there is a linear transformation J between the space of observables to the space of generators called ‘dynamic correspondence’ [9] or “the equivalence of observables and generators” [8] such that $J\sigma J = \sqrt{x}I$.

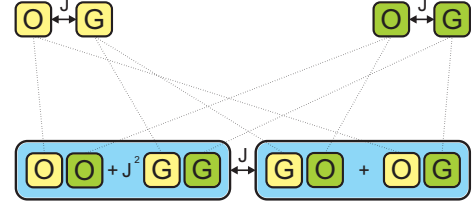


FIG. 3. Composability of the observables and generators map.

This map is the root cause of the existence of complex numbers in all quantum mechanics formulations even when quantum mechanics is represented over real numbers [13]. The classical case is subtle and it looks like dynamic correspondence is not present, but in fact in all composability classes one can construct an associative product based on the products α and σ and in the case of classical physics J is a nilpotent element.

$$\begin{aligned} J &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & J^2 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 & \text{Hyperbolic} \\ & & & & \text{Not present in nature} \\ J &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & J^2 &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0 & \text{Parabolic} \\ & & & & \text{Classical Mechanics} \\ J &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & J^2 &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -1 & \text{Elliptic} \\ & & & & \text{Quantum Mechanics} \end{aligned}$$

FIG. 4. The three composability classes map.

We can now investigate the consequences of the skew-symmetry of the product α over the product σ .

In general the products α and σ are not necessarily associative. for an arbitrary product $*$, a measure of non-associativity is the associator:

$$[f, g, h]_* = (f * g) * h - f * (g * h) \quad (17)$$

Using the Jordan and Leibnitz identities along with the skew-symmetry of the product α , one can show [5] that there is a relationship between the α associator and σ associator, called the Petersen identity:

$$[f, g, h]_\sigma + J[f, g, h]_\alpha = 0 \quad (18)$$

In turn this means that an associative product β can be introduced as: $\beta = \sigma + J\alpha$.

When $J \neq 0$, in the quantal case, by choosing $f = h$ and $f = h\sigma h$ in Eq. (18) and using the Leibnitz and skew-symmetry property of the product α we obtain: $[h, g, h]_\sigma = 0$ and $[h\sigma h, g, h]_\sigma = 0$. This means that the product σ obeys the flexible law and Jordan identity [5].

For the classical case, the product σ is always an associative product in addition to being commutative (symmetric).

A direct consequence of those results implies that quantum mechanics cannot be formulated over octonions because the product β has to be associative and therefore the Jordan algebra of observables σ cannot be special. The classification of real Jordan algebras restricts the allowed number systems for quantum mechanics.

We can also confirm that the tensor product is also associative by verifying that: $\alpha_{(12)3} = \alpha_{1(23)}$ and $\sigma_{(12)3} = \sigma_{1(23)}$ by direct application of Eq. (12). Similarly we can see that the tensor product is also commutative: $\alpha_{(12)} = \alpha_{(21)}$ and $\sigma_{(12)} = \sigma_{(21)}$, thus making the composability category a commutative monoid.

One of the composability classes, the hyperbolic case, can be eliminated by requiring having a non-contextual assignment of truth to statements about nature. This captures the idea of objective reality and allows physics to be an experimental science.

If in quantum mechanics one changes the imaginary unit of complex numbers from $\sqrt{-1}$ to $\sqrt{+1} \neq \pm 1$, one obtains the so called “hyperbolic quantum mechanics” over split-complex numbers [14]. This corresponds to the hyperbolic composability case, and this quantal case violates the Stone-von Neumann theorem because it has non-equivalent representations even in the finite dimensional case. As such predictions of this theory depend on whether one chooses the position or the momentum representation. Therefore the only physical cases remaining are classical mechanics with the product α the Poisson bracket, and the product σ the point-wise function multiplication, and quantum mechanics with the product α the commutator and the product σ the anti-commutator. By composability, nature can only be in one composability class, and the way in which we can distinguish in nature between the elliptic and parabolic composability classes is by experimental evidence.

It is well known that classical mechanics obeys Bell inequalities [2] and quantum mechanics can achieve higher correlations. The strong experimental evidence in favor of violations of Bell inequalities starts with the Aspect experiment [15].

So far we had discussed only about the Jordan and Lie algebras of quantum mechanics. The next step is to introduce a norm and the concept of a Banach space.

The Jordan algebra of observables is the self-adjoint part of a C^* algebra obeying the property:

$$\|x^*x\| = \|x\|^2 \quad (19)$$

with

$$\|x\|^2 = \text{Spectral radius of } x^*x \quad (20)$$

Given the spectral radius, there is no freedom in choosing this norm [6] and now we can use the standard GNS

construction [7] to construct a representation of a C^* algebra as the algebra of bounded operators on a Hilbert space, arriving at the usual formulation of quantum mechanics.

The algebraic formalism of quantum mechanics [16, 17] is a well established domain and we are only providing a justification of the entry point into this large domain.

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SUPPLEMENTARY INFORMATION

This document contains the following sections:

- I. Existence of a second product
- II. Composability requirements on two products
- III. Symmetric and skew-symmetric decompositions
- IV. Reduced composability equations
- V. Deformation quantization
- VI. Norm, spectrum, involution, and non-contextuality

EXISTENCE OF A SECOND PRODUCT

Here we show that it is impossible to have only one nontrivial product in the composability category. We start from the existence of a unit element for the tensor product and we pick $1 \in \mathbb{R}$ understood as a constant function. The existence of a unit demands:

$$(f \otimes 1)\rho_{12}(g \otimes 1) = (f\rho g) \otimes 1 = (f\rho g) \quad (21)$$

with ρ_{12} the product ρ in a bipartite case. Invariance of the laws of nature under composability which demands the bipartite products to be built out of the products listed in the composability category.

Supposing that only a product ρ exists, invariance of the laws of nature under composability demands that the bipartite product ρ_{12} must be of the form:

$$(f \otimes 1)\rho_{12}(g \otimes 1) = a(f\rho g) \otimes (1\rho 1) \quad (22)$$

but this is zero because $(1\rho 1) = 0$ from the Leibniz identity.

COMPOSABILITY REQUIREMENTS ON TWO PRODUCTS

Let us now consider four functions f_1, f_2, g_1, g_2 over the manifold M at a point p . Let us consider an additional product θ . By invariance under composability, the most general way to construct the products ρ and θ in a bipartite system is as follows:

$$\begin{aligned} (f_1 \otimes f_2)\rho_{12}(g_1 \otimes g_2) = & a(f_1\rho g_1) \otimes (f_2\rho g_2) + \\ & b(f_1\rho g_1) \otimes (f_2\theta g_2) + c(f_1\theta g_1) \otimes (f_2\rho g_2) + \\ & d(f_1\theta g_1) \otimes (f_2\theta g_2) \end{aligned} \quad (23)$$

$$\begin{aligned} (f_1 \otimes f_2)\theta_{12}(g_1 \otimes g_2) = & x(f_1\rho g_1) \otimes (f_2\rho g_2) + \\ & y(f_1\rho g_1) \otimes (f_2\theta g_2) + z(f_1\theta g_1) \otimes (f_2\rho g_2) + \\ & w(f_1\theta g_1) \otimes (f_2\theta g_2) \end{aligned} \quad (24)$$

The strategy is to use the existence of the unit of the composability category to determine the coefficients a, b, c, d, x, y, z, w . We also want to normalize the definition of product θ such that $1\theta 1 = 1$.

Following Grgin's approach [1], we pick the constant functions $f_1 = g_1 = 1 \in \mathbb{R}$ while using $\mathbb{R} \otimes \mathcal{U} = \mathcal{U}$ and $1\rho 1 = 0$.

Under this substitution, in ρ_{12} only terms corresponding to the c and d coefficients survive and this demands $c = 1$ and $d = 0$.

Similarly, for θ_{12} this demands $z = 0$ and $w = 1$.

Doing the same thing by picking $f_2 = g_2 = 1 \in \mathbb{R}$ results in $b = 1$ and $y = 0$.

In shorthand notation:

$$\rho_{12} = \rho\theta + \theta\rho + a\rho\rho \quad (25)$$

and

$$\theta_{12} = \theta\theta + x\rho\rho \quad (26)$$

Now we will prove that $a = 0$. To do this we will use the Leibniz identity on a bipartite system:

$$\begin{aligned} (f_1 \otimes f_2)\rho_{12}[(g_1 \otimes g_2)\rho_{12}(h_1 \otimes h_2)] = & \\ [(f_1 \otimes f_2)\rho_{12}(g_1 \otimes g_2)]\rho_{12}(h_1 \otimes h_2) + & \\ (g_1 \otimes g_2)\rho_{12}[(f_1 \otimes f_2)\rho_{12}(h_1 \otimes h_2)] & \end{aligned} \quad (27)$$

substituting the expression for ρ_{12} and tracking only the “ a ” terms meaning ignoring any terms involving the θ product (because ρ is a linear product) we obtain:

$$\begin{aligned} a^2[f_1\rho(g_1\rho h_1)] \otimes [f_2\rho(g_2\rho h_2)] = & \\ a^2[(f_1\rho g_1)\rho h_1] \otimes [(f_2\rho g_2)\rho h_2] + & \\ a^2[g_1\rho(f_1\rho h_1)] \otimes [g_2\rho(f_2\rho h_2)] & \end{aligned} \quad (28)$$

Applying the Leibniz identity again on the right hand side and canceling terms yields:

$$\begin{aligned} a^2\{[(f_1\rho g_1)\rho h_1] \otimes [g_2\rho(f_2\rho h_2)] + & \\ [g_1\rho(f_1\rho h_1)] \otimes [(f_2\rho g_2)\rho h_2]\} = 0 & \end{aligned} \quad (29)$$

which is valid for all f_1, f_2, g_1, g_2 and hence $a = 0$.

SYMMETRIC AND SKEW-SYMMETRIC DECOMPOSITIONS

We observe that if ρ is a skew-symmetric product and θ is a symmetric product the symmetry and skew-symmetry is maintained under composition. This suggests decomposing ρ and θ as follows:

$$\begin{aligned}\rho &= \alpha + \tau \\ \theta &= \sigma + \pi\end{aligned}\tag{30}$$

with:

$$\begin{aligned}f\alpha g &= (f\rho g - g\rho f)/2 \\ f\tau g &= (f\rho g + g\rho f)/2 \\ f\sigma g &= (f\theta g + g\theta f)/2 \\ f\pi g &= (f\theta g - g\theta f)/2\end{aligned}\tag{31}$$

The new products have well defined symmetry properties and because we have both a left and right Leibniz identity, those properties get inherited by the products α and τ . Therefore the following properties hold:

$$\begin{aligned}1\alpha 1 &= 0 \\ 1\tau 1 &= 0 \\ 1\pi 1 &= 0 \\ 1\sigma 1 &= 1\end{aligned}\tag{32}$$

The most general odd and even bipartite products are:

$$\begin{aligned}\sigma_{12} &= a^1\sigma_1\sigma_2 + a^2\sigma_1\tau_2 + a^3\tau_1\sigma_2 + a^4\tau_1\tau_2 + \\ &\quad a^5\pi_1\pi_2 + a^6\pi_1\alpha_2 + a^7\alpha_1\pi_2 + a^8\alpha_1\alpha_2 \\ \pi_{12} &= a^9\pi_1\sigma_2 + a^{10}\sigma_1\pi_2 + a^{11}\pi_1\tau_2 + a^{12}\tau_1\pi_2 + \\ &\quad a^{13}\alpha_1\tau_2 + a^{14}\tau_1\alpha_2 + a^{15}\alpha_1\sigma_2 + a^{16}\sigma_1\alpha_2 \\ \alpha_{12} &= a^{17}\pi_1\sigma_2 + a^{18}\sigma_1\pi_2 + a^{19}\pi_1\tau_2 + a^{20}\tau_1\pi_2 + \\ &\quad a^{21}\alpha_1\tau_2 + a^{22}\tau_1\alpha_2 + a^{23}\alpha_1\sigma_2 + a^{24}\sigma_1\alpha_2 \\ \tau_{12} &= a^{25}\sigma_1\sigma_2 + a^{26}\sigma_1\tau_2 + a^{27}\tau_1\sigma_2 + a^{28}\tau_1\tau_2 \\ &\quad + a^{29}\pi_1\pi_2 + a^{30}\pi_1\alpha_2 + a^{31}\alpha_1\pi_2 + a^{32}\alpha_1\alpha_2\end{aligned}\tag{33}$$

Proceeding similarly and using the existence of the composability unit element, let us choose $f_1 = f_2 = 1$ and use this on product ρ (and not on the products α or τ individually to avoid introducing spurious constraints):

$$(1 \otimes 1)\rho_{12}(g_1 \otimes g_2) = g_1\rho g_2\tag{34}$$

This demands $a^{18} = a^{25} = 0$ and $a^{24} = a^{26} = 1$ if their corresponding products are not identically zero.

Similarly, by picking $g_1 = g_2 = 1$ this demands $a^{17} = 0$ and $a^{23} = a^{27} = 1$. Picking $f_2 = g_1 = 1$ yields $a^{29} = 0$.

Now we can do a similar argument on product θ and if we choose $f_1 = f_2 = 1$, this demands $a^2 = a^{16} = 0$ and

$a^1 = a^{10} = 1$. Also if we choose $g_1 = g_2 = 1$ this implies $a^9 = 1$ and $a^3 = a^{15} = 0$.

Finally, demanding that $\rho_{12} = \rho_1\theta_2 + \theta_1\rho_2$ results in: $a^{23} = a^{31} = a^{20} = a^{24} = a^{30} = a^{19} = 1$ and $a^{21} = a^{22} = a^{28} = a^{32} = 0$.

Also demanding that $\theta_{12} = \theta_1\theta_2 + x\rho_1\rho_2$ yields: $a^5 = 1$, $a^4 = a^8 = a^{13} = a^{14} = x$, and $a^6 = a^7 = a^{11} = a^{12} = 0$.

Collecting all those results above demands the following relationships:

$$\begin{aligned}\sigma_{12} &= \sigma_1\sigma_2 + x\tau_1\tau_2 + \pi_1\pi_2 + x\alpha_1\alpha_2 \\ \alpha_{12} &= \pi_1\tau_2 + \tau_1\pi_2 + \alpha_1\sigma_2 + \sigma_1\alpha_2 \\ \tau_{12} &= \sigma_1\tau_2 + \tau_1\sigma_2 + \pi_1\alpha_2 + \alpha_1\pi_2 \\ \pi_{12} &= \pi_1\sigma_2 + \sigma_1\pi_2 + x\alpha_1\tau_2 + x\tau_1\alpha_2\end{aligned}\tag{35}$$

REDUCED COMPOSABILITY EQUATIONS

Let us now investigate what happens when one of the products in the above equations is identically zero.

We start by assuming that $\tau_1 = \tau_2 = 0$ as we compose two one dimensional systems. From α_{12} and π_{12} we see that α and π are now proportional. Substituting this back in τ_{12} results in τ_{12} being proportional with $\alpha\alpha$ which we have shown earlier to be untenable under composability if we want non-trivial solutions, so $\tau \equiv 0$ for all systems. This in turns means that $\pi\alpha + \alpha\pi = 0$. One α term can be eliminated by picking one argument to be 1, and then either of the product must be identically zero. If $\alpha \equiv 0$, then the product ρ is trivial and we are not interested in this case. The only remaining alternative is to have $\pi \equiv 0$ and hence the products ρ, θ are reducing themselves to α, σ .

Suppose now $\alpha \equiv 0$ instead. By a similar argument, ρ, θ are reducing themselves to τ, σ .

Therefore the final composability results are either:

$$\begin{aligned}\sigma_{12} &= \sigma_1\sigma_2 + x\alpha_1\alpha_2 \\ \alpha_{12} &= \alpha_1\sigma_2 + \sigma_1\alpha_2\end{aligned}\tag{36}$$

or

$$\begin{aligned}\sigma_{12} &= \sigma_1\sigma_2 + x\tau_1\tau_2 \\ \tau_{12} &= \tau_1\sigma_2 + \sigma_1\tau_2\end{aligned}\tag{37}$$

with $x = -1, 0, 1$.

DEFORMATION QUANTIZATION

From the Poisson bracket, let us define an operator ∇ as follows:

$$\overleftrightarrow{\nabla} = \sum_{i=1}^N \left[\overleftarrow{\frac{\partial}{\partial x_i} \frac{\partial}{\partial p_i}} - \overrightarrow{\frac{\partial}{\partial p_i} \frac{\partial}{\partial x_i}} \right] \quad (38)$$

which realizes the product α for classical mechanics: $f\alpha g = f \overleftrightarrow{\nabla} g$.

From Eq. (36), when $x = -1$, one can see a simple realization of it if the products α were a sine function, and the product σ a cosine function of some sort. From correspondence principle, expanding the sine function in Taylor series, and remembering that $-x = \hbar^2/4$ is a small parameter, the natural α and σ realization is:

$$\begin{aligned} \alpha &= \sin \overleftrightarrow{\nabla} \\ \sigma &= \cos \overleftrightarrow{\nabla} \end{aligned} \quad (39)$$

The product α is nothing but the Moyal bracket [2] in deformation quantization for the phase-space formulation of quantum mechanics.

Then one creates an associative product β which in this context is called “the star product” $*$:

$$f * g = f\sigma g + Jf\alpha g = fe^{J\overleftrightarrow{\nabla}}g \quad (40)$$

based on the number system $\mathbb{R} \oplus J\mathbb{R}$ which is the field of complex numbers \mathbb{C} .

Now here are more advanced topics on this subject. The star product can be understood as an infinite sum of terms proportional with the powers of the Planck constant \hbar . Also the star product being associative by construction, associativity transfers to each term in all Planck constant power terms.

A natural question to ask is the equivalence of two star products which is defined as a morphism which in turn can be decomposed as an infinite sum of morphisms proportional with a given power of the Planck constant.

The equivalence classes of star products on symplectic manifolds are in one-to-one correspondence with second de Rham cohomology $H_{dR}^2(M)$. Therefore there could be inequivalent ways of quantization, and hence quantization is not a functor. There are several rigorous quantization strategies available, like for example geometric quantization [3], but quantization is a hard area in general when one departs from the simplistic cases which work using Dirac prescription of replacing the Poisson bracket with the commutator.

In the phase space formulation, quantum mechanics needs to also employ Wigner’s quasiprobabilities [4] which generalize positive probabilities into negative territory. This is also a subtle area because negative probabilities do not necessarily mean quantum behavior [5]. Quantum mechanics can be put in the phase space formalism, and also classical mechanics can be cast in the Hilbert space formalism as well [6], but in both cases

there are naturalness problems. Existence of quasiprobabilities signals that there are better mathematical realizations of the composability equations in the quantum mechanics case and indeed such realizations exist.

NORM, SPECTRUM, INVOLUTION, AND NON-CONTEXTUALITY

The main result of this paper can be understood as a fixed point theorem stemming from the invariance of the laws of nature. Quantum mechanics is one of the three possible “fixed points” and its structure is rigidly defined by the constraints originating from this with no freedom to be any other way. Norm, spectrum, and involution are usually studied as separate axioms but they are consequences of the algebraic constraints of elliptic composability.

Norm can be expressed algebraically because:

$$\|x\| = \sup\{|\lambda| : x - \lambda 1 \text{ is not invertible}\} \quad (41)$$

With the advantage of known results [7–9] clarifying the conditions under which state spaces, C^* algebras, and normed Jordan algebras are related, we can start from the associative, Jordan, and Lie algebras together with dynamic correspondence and prove it must be a C^* algebras instead of the usual way of postulating the norm and the C^* property as a starting point.

The root cause of involution is the “dynamic correspondence” or our map J . An element is called hermitean if it is invariant under involution and any element x of the associative algebra can be uniquely decomposed as $x = x_1 + Jx_2$ with x_1, x_2 hermitean elements. From this it is easy to show that all spectral values are real.

The hyperbolic composability case is subtle because there are cases where the spectrum contains square roots and in this case there is an ambiguity to consider the square roots real or imaginary. Since in this case the vector space is over the split complex numbers, the fundamental theorem of algebra does not hold. Also the C^* condition does not hold either, and a straight generalization of the C^* algebra as an algebra over split complex numbers is not possible.

The hyperbolic composability case was rejected by an appeal to non-contextuality but several other approaches can be taken to reject this non-physical case. Non-contextuality was chosen because non-contextuality of probability is an assumption in Gleason’s theorem [10] which establishes the Born rule. Hence non-contextuality completes the recovery of the standard formalism of quantum mechanics.

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